

Joint Graph Decomposition and Node Labeling by Local Search

Evgeny Levinkov Siyu Tang Eldar Insafutdinov Bjoern Andres

Max Planck Institute for Informatics, Saarbrücken, Germany

Abstract

We state a combinatorial optimization problem whose feasible solutions define both a decomposition and a node labeling of a given graph. This problem offers a common mathematical abstraction of seemingly unrelated computer vision tasks, including instance-separating semantic segmentation, articulated human body pose estimation and multiple object tracking. Conceptually, the problem we propose generalizes the unconstrained integer quadratic program and the minimum cost lifted multicut problem, both of which are NP-hard. In order to find feasible solutions efficiently, we define a local search algorithm that converges monotonously to a local optimum, offering a feasible solution at any time. To demonstrate the effectiveness of this algorithm in solving computer vision tasks, we report running times and competitive solutions for two above-mentioned applications.

1. Introduction and Related Work

In this article, we state a combinatorial optimization problem whose feasible solutions define both a decomposition and a node labeling of a given graph (Fig. 1). This problem offers a common mathematical abstraction of seemingly unrelated computer vision tasks:

Multiple object tracking [2, 3, 4, 7, 13, 17, 19, 29, 30] can be seen as the task of deciding, for every point in an image, whether this point depicts an object, and of deciding, for every pair of points that depict objects, if the object is the same. Tang et al. [26, 27] abstract this task as a graph decomposition and node labeling problem w.r.t. a finite graph whose nodes are parts of the image, and w.r.t. 01-labels indicating that a part depicts an object. We generalize their problem to more labels and more complex objective functions.

Instance-separating semantic segmentation [5, 6, 15, 22, 23, 24, 31, 32] is the task of assigning, to every point in an image, a label that identifies a class of objects (e.g., human, car, bicycle), and of deciding, for every pair of points, whether they belong to the same object. Kroeger et al. [14] state this problem as a graph decomposition and node la-

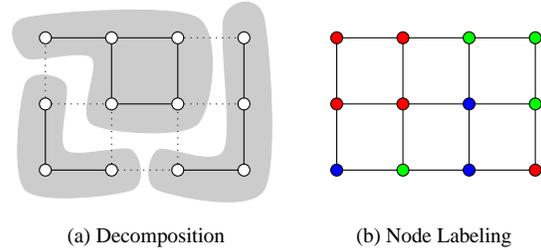


Figure 1: This article studies an optimization problem whose feasible solutions define both a decomposition (a) and a node labeling (b) of a given graph $G = (V, E)$. A decomposition of G is a partition Π of the node set V such that, for every $V' \in \Pi$, the subgraph of G induced by V' is connected. A node labeling of G is a map $f : V \rightarrow L$ from its node set V to a finite, non-empty set L of labels.

belonging problem w.r.t. a (super)pixel adjacency graph of the image. Feasible solutions of their problem have three properties: For each object, all its pixels have the same label. Pixels with distinct labels belong to distinct objects. Pixels with the same label, including neighboring pixels, can belong to distinct objects. We generalize their problem to relaxations of these constraints and more complex objective functions.

Articulated human body pose estimation [20, 9] is the task of deciding, for every point in an image, whether this point depicts a part of the human body, and of deciding, for every pair of points that depict body parts, if they belong to the same body. Pishchulin et al. [20] and Insafutdinov et al. [9] abstract this problem as a graph decomposition and node labeling problem w.r.t. a finite graph whose nodes are putative detections of body parts and w.r.t. labels that identify body parts (head, wrist, etc.) and background. We generalize their problem to more complex objective functions.

Formally, the minimum cost node labeling lifted multicut problem we propose and refer to as the NL-LMP generalizes the NP-hard unconstrained integer quadratic program, UIQP, that has been studied intensively in the context of graphical models [10] and also generalizes the NP-hard minimum cost lifted multicut problem [12], LMP. Unlike in pure node la-

being problems such as the UIQP, neighboring nodes with the same label can be assigned to distinct components by feasible solutions of the NL-LMP, and neighboring nodes with distinct labels can be assigned to the same component. Unlike in pure decomposition problems such as the LMP, the cost of assigning nodes to the same component or distinct components can depend on node labels. Also unlike in the LMP, constraining nodes with the same label to the same component constrains the feasible decompositions to be k -colorable, with $k \in \mathbb{N}$ the number of labels. For $k = 2$ in particular, the NL-LMP specializes to the MAX-CUT problem.

In order to find feasible solutions of the NL-LMP efficiently, we define and implement two local search algorithms that converges monotonously to a local optimum, offering a feasible solution at any time. These algorithms does not compute lower bounds. Their feasible solutions come without approximation certificates. Hence, they belong to the class of primal feasible heuristics for the NL-LMP. The first algorithm we define as a baseline, alternating Kernighan-Lin search with joins and node relabeling, KLj-r, is a straightforward generalization of the algorithm KLj defined by Keuper et al. [12]. In fact, our implementation is an extension of their C++ code. The second algorithm we define as our main contribution, joint Kernighan-Lin search with joins and node relabeling, KLj+r, is a non-straightforward generalization of KLj that combines updates of the decomposition with updates of the node labeling in a novel manner. Both algorithms build on seminal work of Kernighan and Lin [11].

To demonstrate the effectiveness of the algorithms in solving computer vision tasks, we analyze their absolute running time and output for instances of the NL-LMP for two above-mentioned applications. *We will make the code and data publicly available upon acceptance of the paper.*

2. Problem

In this section, we define the minimum cost node labeling lifted multicut problem (NL-LMP). Sections 2.1–2.3 offer an intuition for its parameters, feasible solutions and cost function. Section 2.4 offers a concise and rigorous definition. Section 2.5 discusses special cases.

2.1. Parameters

Any instance of the NL-LMP is defined with respect to the following parameters:

- A connected graph $G = (V, E)$ whose decompositions we care about, e.g., the pixel grid graph of an image.
- A graph $G' = (V, E')$ with $E \subseteq E'$. This graph can contain as edges pairs of nodes that are not neighbors in G . It defines the structure of the cost function.

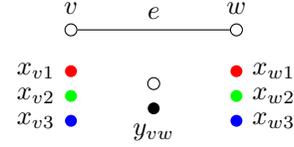


Figure 2: Every feasible solution of the NL-LMP is a pair (x, y) of 01-vectors $x \in \{0, 1\}^{V \times L}$ and $y \in \{0, 1\}^{E'}$. More specifically, x is constrained such that, for every node $v \in V$, there is precisely one label $l \in L$ such that $x_{vl} = 1$. y is constrained so as to well-define a decomposition of G by the set $\{e \in E \mid y_e = 1\}$ of those edges that straddle distinct components.

- A digraph $H = (V, A)$ that fixes an arbitrary orientation of the edges E' . That is, for every edge $\{v, w\}$ of G' , the graph H contains either the edge (v, w) or the edge (w, v) . Moreover, H does not contain additional edges. Formally, H is such that for all $v, w \in V$:

$$\{v, w\} \in E' \Leftrightarrow vw \in A \vee (w, v) \in A \quad (1)$$

$$(v, w) \notin A \vee (w, v) \notin A \quad (2)$$

- A finite, non-empty set L called the set of (node) labels
- The following functions whose values are called costs:
 - $c : V \times L \rightarrow \mathbb{R}$. For any node $v \in V$ and any label $l \in L$, the cost c_{vl} is payed iff v is labeled l .
 - $c^\sim : A \times L^2 \rightarrow \mathbb{R}$. For any edge $vw \in A$ and any labels $ll' \in L^2$, the cost $c_{vw, ll'}^\sim$ is payed iff v is labeled l and w is labeled l' and v and w are in the same component.
 - $c^\not\sim : A \times L^2 \rightarrow \mathbb{R}$. For any edge $vw \in A$ and any labels $ll' \in L^2$, the cost $c_{vw, ll'}^\not\sim$ is payed iff v is labeled l and w is labeled l' and v and w are in distinct components.

2.2. Feasible Set

Every feasible solution of the NL-LMP is a pair (x, y) of 01-vectors $x \in \{0, 1\}^{V \times L}$ and $y \in \{0, 1\}^{E'}$; see Fig. 2. More specifically, x is constrained such that, for every node $v \in V$, there is precisely one label $l \in L$ such that $x_{vl} = 1$. y is constrained so as to well-define a decomposition of G by the set $\{e \in E \mid y_e = 1\}$ of those edges that straddle distinct components. Formally, $(x, y) \in X_{VL} \times Y_{GG'}$ with X_{VL} and $Y_{GG'}$ defined below.

- $X_{VL} \subseteq \{0, 1\}^{V \times L}$, the set of all characteristic functions of maps from V to L , i.e., the set of all $x \in \{0, 1\}^{V \times L}$ such that

$$\forall v \in V : \sum_{l \in L} x_{vl} = 1 \quad (3)$$

For any $x \in X$, any $v \in V$ and any $l \in L$ with $x_{vl} = 1$, we say that node v is *labeled* l by x .

- $Y_{GG'} \subseteq \{0, 1\}^{E'}$, the set of all characteristic functions of multicuts of G' lifted from G . For any $y \in Y_{GG'}$ and any $e = \{v, w\} \in E'$, $y_e = 1$ indicates that v and w are in distinct components of the decomposition of G defined by the multicut $\{e' \in E \mid y_{e'} = 1\}$ of G . Formally, $Y_{GG'}$ is the set of all $y \in \{0, 1\}^{E'}$ that satisfy the following system of linear inequalities:

$$\forall C \in \text{cycles}(G) \forall e \in C : y_e \leq \sum_{e' \in C \setminus \{e\}} y_{e'} \quad (4)$$

$$\forall \{v, w\} \in E' \setminus E \forall P \in vw\text{-paths}(G) : y_{\{v, w\}} \leq \sum_{e \in P} y_e \quad (5)$$

$$\forall \{v, w\} \in E' \setminus E \forall C \in vw\text{-cuts}(G) : 1 - y_{\{v, w\}} \leq \sum_{e \in C} (1 - y_e) . \quad (6)$$

2.3. Cost Function

For every $x \in \{0, 1\}^{V \times L}$ and every $y \in \{0, 1\}^{A \times L^2}$, a cost $\varphi(x, y) \in \mathbb{Z}$ is defined by the form

$$\begin{aligned} \varphi(x, y) = & \sum_{v \in V} \sum_{l \in L} c_{vl} x_{vl} \\ & + \sum_{vw \in A} \sum_{l, l' \in L^2} c_{vw, ll'}^{\sim} x_{vl} x_{wl'} (1 - y_{\{v, w\}}) \\ & + \sum_{vw \in A} \sum_{l, l' \in L^2} c_{vw, ll'}^{\not\sim} x_{vl} x_{wl'} y_{\{v, w\}} . \quad (7) \end{aligned}$$

2.4. Definition

We define the NL-LMP rigorously and concisely in the form of a linearly constrained binary cubic program.

Definition 1 For any connected graph $G = (V, E)$, any graph $G' = (V, E')$ with $E \subseteq E'$, any orientation $H = (V, A)$ of G' , any finite, non-empty set L , any function $c : V \times L \rightarrow \mathbb{Z}$ and any functions $c^{\sim}, c^{\not\sim} : A \times L^2 \rightarrow \mathbb{Z}$, the instance of the *minimum cost node-labeling lifted multicut problem* (NL-LMP) with respect to $(G, G', H, L, c, c^{\sim}, c^{\not\sim})$ has the form

$$\min_{(x, y) \in X_{VL} \times Y_{GG'}} \varphi(x, y) . \quad (8)$$

2.5. Special Cases

Below, we show that the NL-LMP generalizes the UIQP. This connects the NL-LMP to work on graphical models with second-order functions and finitely many states. In addition, we show that NL-LMP generalizes the LMP, connecting the NL-LMP to recent work on lifted multicuts. Finally, we show that the NL-LMP is general enough for subgraph selection, connectedness and disconnectedness constraints.

2.5.1 Unconstrained Integer Quadratic Program

Definition 2 For any graph $G' = (V, E')$, any orientation $H = (V, A)$ of G' , any finite, non-empty set L , any $c : V \times L \rightarrow \mathbb{Z}$ and any $c' : A \times L^2 \rightarrow \mathbb{Z}$, the instance of the UIQP with respect to (G', H, L, c, c') has the form

$$\min_{x \in X_{VL}} \sum_{v \in V} \sum_{l \in L} c_{vl} x_{vl} + \sum_{vw \in A} \sum_{l, l' \in L^2} c'_{vw, ll'} x_{vl} x_{wl'} . \quad (9)$$

Lemma 1 For any graph $G' = (V, E')$, any instance (G', H, L, c, c') of the UIQP and any $x \in X_{VL}$, x is a solution of this instance of the UIQP iff $(x, 1_{E'})$ is a solution of the instance $(G', G', H, L, c, c', c')$ of the NL-LMP.

PROOF Without loss of generality, we can assume that G' is connected. (Otherwise, add edges between nodes $v, w \in V$ as necessary and set $c'_{vw, ll'} = 0$ for any $l, l' \in L$.)

For any $x \in X_{GL}$, the pair $(x, 1_{E'})$ is a feasible solution of the instance of the NL-LMP because the map $1_{E'} : E' \rightarrow \{0, 1\} : e \mapsto 1$ is such that $1_{E'} \in Y_{G'G'}$.

Moreover, $(x, 1_{E'})$ is a solution of the instance of the NL-LMP iff x is a solution of the instance of the UIQP because, for $c^{\not\sim} = c^{\sim}$, the form (7) of the cost function of the NL-LMP specializes to the form (9) of the cost function of the UIQP.

2.5.2 Minimum Cost Lifted Multicut Problem

Definition 3 [1] For any connected graph $G = (V, E)$, any graph $G' = (V, E')$ with $E \subseteq E'$ and any $c' : E' \rightarrow \mathbb{Z}$, the instance of the minimum cost lifted multicut problem (LMP) with respect to (G, G', c') has the form

$$\min_{y \in Y_{GG'}} \sum_{e \in E'} c'_e y_e . \quad (10)$$

Lemma 2 Let (G, G', c') be any instance of the LMP. Let $(G, G', H, L, c, c^{\sim}, c^{\not\sim})$ be the instance of the NL-LMP with the same graphs and such that

$$L = \{1\} \quad c = 0 \quad c^{\sim} = 0 \quad (11)$$

$$\forall (v, w) \in A : c_{vw, 11}^{\not\sim} = c'_{\{v, w\}} . \quad (12)$$

Then, for any $y \in \{0, 1\}^{E'}$, y is a solution of the instance of the LMP iff $(1_{V \times L}, y)$ is a solution of the instance of the NL-LMP.

PROOF Trivially, y is a feasible solution of the instance of the LMP iff $(1_{V \times L}, y)$ is a feasible solution of the instance of the NL-LMP. More specifically, y is a solution of the instance of the LMP iff $(1_{V \times L}, y)$ is a solution of the instance of the NL-LMP because, for any $x \in X_{VL}$, the cost function

(7) of the NL-LMP assumes the special form below which is identical with the form in (10).

$$\varphi(x, y) \stackrel{(3),(11)}{=} \sum_{vw \in A} c_{vw,11}^{\mathcal{L}} y_{\{v,w\}} \stackrel{(12)}{=} \sum_{e \in E'} c_e' y_e . \quad (13)$$

2.5.3 Subgraph Selection

Applications such as [9, 20, 26, 27], require us to not only decompose a graph and label its nodes but to also select a subgraph. The NL-LMP is general enough to model subgraph selection. To achieve this, one proceeds in two steps: Firstly, one introduces a special label $\epsilon \in L$ to indicate that a node is not an element of the subgraph. We call these nodes *inactive*. All other nodes are called *active*. Secondly, one chooses a large enough $c^* \in \mathbb{N}$, a $c^\dagger \in \mathbb{N}_0$ and $c^\sim, c^\mathcal{L}$ such that

$$\forall vw \in A \forall l \in L \setminus \{\epsilon\} : c_{vw,l\epsilon}^\sim = c_{vw,\epsilon l}^\sim = c^* \quad (14)$$

$$c_{vw,l\epsilon}^\mathcal{L} = c_{vw,\epsilon l}^\mathcal{L} = 0 \quad (15)$$

$$\forall vw \in A : c_{vw,\epsilon\epsilon}^\sim = c^\dagger . \quad (16)$$

By (14), inactive nodes are not joined with active nodes in the same component. By (15), cutting an inactive node from an active node has zero cost. By (16), joining inactive nodes has cost c^\dagger , possibly zero. Choosing c^\dagger large enough implements an additional constraint proposed in [26] that inactive nodes are necessarily isolated. It is by this constraint and by a two-elementary label set that [26] is a specialization of the NL-LMP.

2.5.4 (Dis-)Connectedness Constraints

Some applications require us to constrain certain nodes to be in distinct components. One example is instance-separating semantic segmentation where nodes with distinct labels necessarily belong to distinct segments. Other applications require us to constrain certain nodes to be in the same component. One example is articulated human body pose estimation for a single human in the optimization framework of [20] where every pair of active nodes necessarily belongs to the same human. Yet another example is connected foreground segmentation [18, 21, 25, 28] in which every pair of distinct foreground pixels necessarily belongs to the same segment.

The NL-LMP is general enough to model a combination of connectedness constraints and disconnectedness constraints. In order to constrain distinct nodes $v, w \in V$ with labels $l, l' \in L$ to be in *the same component*, one introduces an edge $(v, w) \in A$, a large enough $c^* \in \mathbb{N}$ and costs c^\sim such that $c_{vw,ll'}^\sim = c_{vw,l'l}^\sim = c^*$. In order to constrain distinct nodes $v, w \in V$ with labels $l, l' \in L$ to be in *distinct components*, one introduces an edge $(v, w) \in A$, a large

enough $c^* \in \mathbb{N}$ and costs $c^\mathcal{L}$ such that $c_{vw,ll'}^\mathcal{L} = c_{vw,l'l}^\mathcal{L} = c^*$.

3. Algorithms

In this section, we define two local search algorithms that compute feasible solutions of the NL-LMP efficiently. Both algorithms attempt to improve a current feasible solution recursively by *transformations*. One class of transformations alters the node labeling of the graph by replacing a single node label. A second class of transformations alters the decomposition of the graph by moving a single node from one component to another. A third class of transformations alters the decomposition of the graph by joining two components.

As proposed by Kernighan and Lin [11] and applied to the LMP by Keuper et al. [12], a local search is carried out not over the set of individual transformations of the current feasible solution but over a set of sequences of transformations. Complementary to this idea, we define and implement two schemes of combining transformations of the decomposition of the graph with transformations of the node labeling of the graph. This leads us to define two local search algorithms for the NL-LMP.

3.1. Encoding Feasible Solutions

To encode feasible solutions $(x, y) \in X_{VL} \times Y_{GG'}$ of the NL-LMP, we consider two maps: A *node labeling* $\lambda : V \rightarrow L$ that defines the $x^\lambda \in X_{VL}$ such that

$$\forall v \in V \forall l \in L : x_{vl}^\lambda = 1 \Leftrightarrow \lambda(v) = l , \quad (17)$$

and a so-called *component labeling* $\mu : V \rightarrow \mathbb{N}$ that defines the $y^\mu \in \{0, 1\}^{E'}$ such that

$$\forall \{v, w\} \in E' : y_{\{v,w\}}^\mu = 0 \Leftrightarrow \mu(v) = \mu(w) . \quad (18)$$

3.2. Transforming Feasible Solutions

To improve feasible solutions of the NL-LMP recursively, we consider three transformations of the encodings λ and μ :

For any node $v \in V$ and any label $l \in L$, the transformation $T_{vl} : L^V \rightarrow L^V : \lambda \mapsto \lambda'$ changes the label of the node v to l , i.e.

$$\forall w \in V : \lambda'(w) := \begin{cases} l & \text{if } w = v \\ \lambda(w) & \text{otherwise} \end{cases} . \quad (19)$$

For any node $v \in V$ and component index $m \in \mathbb{N}$, the transformation $T'_{vm} : \mathbb{N}^V \rightarrow \mathbb{N}^V : \mu \mapsto \mu'$ changes the component index of the node v to m , i.e.

$$\forall w \in V : \mu'(w) := \begin{cases} m & \text{if } w = v \\ \mu(w) & \text{otherwise} \end{cases} . \quad (20)$$

For any component indices $m, m' \in \mathbb{N}$, the transformation $T'_{mm'} : \mathbb{N}^V \rightarrow \mathbb{N}^V : \mu \mapsto \mu'$ puts all nodes currently in the component indexed by m into the component indexed by m' , i.e.

$$\forall w \in V : \mu'(w) := \begin{cases} m' & \text{if } \mu(w) = m \\ \mu(w) & \text{otherwise} \end{cases} . \quad (21)$$

3.3. Repairing Infeasible Points

Not every component labeling μ is such that $y^\mu \in Y_{GG'}$. In fact, y^μ is feasible if and only if, for every $m \in \mu(V)$, the node set $\mu^{-1}(m)$ is connected in G . For efficiency, we allow for transformations (20) whose output μ' violates this condition, as proposed in [12]. This happens when an *articulation node* of a component is moved to a different component. In order to repair any μ' for which $y^{\mu'}$ is infeasible, we consider a map $R : \mathbb{N}^V \rightarrow \mathbb{N}^V : \mu' \mapsto \mu$ such that, for any $\mu' : V \rightarrow \mathbb{N}$ and any distinct $v, w \in V$, we have $\mu(v) = \mu(w)$ if and only if there exists a vw -path in G along which all nodes have the label $\mu'(v)$. We implement R as connected component labeling by breadth-first-search.

3.4. Initializing Feasible Solutions

Initial feasible solutions are given, for instance, by the finest decomposition of the graph G that puts every node in a distinct component, or by the coarsest decomposition of the graph G that puts every node in the same component, each together with any node labeling. We find an initial feasible solution for our local search algorithm by first fixing an optimal label for every node independently and by then solving the resulting LMP, (8) for the fixed labels $x \in X_{VL}$, by means of greedy agglomerative edge contraction [12].

3.5. Searching Feasible Solutions

We now define two local search algorithms that attempt to improve an initial feasible solution recursively, by applying the transformation defined above.

KLj-r Algorithm. The first local search algorithm, alternating Kernighan-Lin search with joins and node relabeling, KLj-r, is a straightforward generalization of the algorithm KLj of [12]. KLj-r alternates between transformations of the labeling and transformations of the decomposition. For a fixed decomposition, the labeling is transformed by Func. 1 which greedily updates labels of nodes independently. For a fixed labeling, the decomposition is transformed by Func. 2, *without those parts of the function that are written in green*. This is precisely the algorithm KLj of [12]. (All symbols that appear in the pseudo-code are defined above, except the iteration counter t , cost differences δ, Δ , and 01-vectors α used for bookkeeping, to avoid redundant operations.)

KLj+r Algorithm. The second local search algorithm, joint Kernighan-Lin search with joins and node relabeling,

Function 1: $(\Delta, \lambda') = \text{update-labeling}(\mu, \lambda)$

$\lambda_0 := \lambda \quad \Delta := 0 \quad t := 0$

repeat

choose $(\hat{v}, \hat{l}) \in \underset{(v,l) \in V \times L}{\text{argmin}} \varphi(x^{T_{vl}(\lambda_t)}, y^{\mu_t}) - \varphi(x^{\lambda_t}, y^{\mu_t})$

$\delta := \varphi(x^{T_{\hat{v}\hat{l}}(\lambda_t)}, y^{\mu_t}) - \varphi(x^{\lambda_t}, y^{\mu_t})$

if $\delta < 0$

$\lambda_{t+1} := T_{\hat{v}\hat{l}}(\lambda_t)$

$\Delta := \Delta + \delta$

$t := t + 1$

else

return (Δ, λ_t)

KLj+r, is a non-straightforward generalization of KLj that combines updates of the decomposition with updates of the node labeling in a novel manner. It is given by Func. 2, *with those parts of the function that are written in green*.

Like the baseline algorithm KLj-r, the algorithm KLj+r occasionally updates the labeling for a fixed decomposition (calls of Func. 1 from Func. 2). Unlike the baseline algorithm KLj-r, the algorithm KLj+r also updates the decomposition and the labeling also jointly. This happens in Func. 3 that is called from KLj+r, *with the part that is written in green*.

Func. 3 looks at two components $V := \mu^{-1}(m)$ and $W := \mu^{-1}(m')$ of the current decomposition. It attempts to improve the decomposition as well as the labeling by moving a node from V to W or from W to V *and by simultaneously changing its label*. As proposed by Kernighan and Lin [11], Func. 3 does not make such transformations greedily but first constructs a sequence of such transformations greedily and then executes the first k in order where k is chosen so as to decrease the objective value maximally. KLj-r construct a sequence of moves analogously, but the node labeling remains fixed throughout every transformation of the decomposition. More generally speaking, KLj+r is a local search algorithm whose local neighborhood is strictly larger than that of KLj-r.

Our C++ implementation computes cost differences incrementally, as proposed in [11], and solves the optimization problem over transformations by means of a priority queue, as proposed in [12]. The time and space complexities are identical to those of KLj and are established analogously. Transformations take linear time in the number of labels but constant time in the size of the graph.

4. Applications

We show applications of the proposed problem and algorithms to two computer vision tasks: articulated human body pose estimation and multiple object tracking. For each task, we set up instances of the NL-LMP w.r.t. published data that we transform only trivially.

4.1. Articulated Human Body Pose Estimation

We turn toward applications of the NL-LMP and the algorithms KLj-r and KLj+r to the task of estimating the articulated poses of all humans visible in an image. Pishchulin et al. [20] and Insafutdinov et al. [9] approach this problem via a graph decomposition and node labeling problem that we identify as a special case of the NL-LMP with $c^{\neq} = 0$ and with subgraph selection (Section 2.5.3). We relate their notation to ours rigorously in the supplement of this paper. Nodes in their graph are putative detections of body parts. Labels define body part classes (head, wrist, etc.). In our notation, $x_{vl} = 1$ indicates that the putative detection v is a body part of class l , and $y_{vw} = 1$ indicates that the body parts v and w belong to distinct humans. The test set of [9] consists of 1758 such instances of the NL-LMP.

To tackle these instances, Insafutdinov et al. define and implement a branch-and-cut algorithm in the integer linear programming software framework Gurobi. We refer to their published C++ implementation as B&C.

Cost and time. In Fig. 3, we compare the convergence of B&C (feasible solutions and lower bounds) with the convergence of our algorithms, KLj-r and KLj+r (feasible solutions only). Shown in this figure is the average objective value over the test set w.r.t. the absolute running time. Thanks to the lower bounds obtained by B&C, it can be seen from this figure that KLj-r and KLj+r arrive at near optimal feasible solutions after 10^{-1} seconds, five orders of magnitude faster than B&C. This result shows that primal feasible heuristics for the NL-LMP, such as KLj-r and KLj+r, are practically useful in the context of this application.

Function 2: $(\Delta', \mu', \lambda') = \text{update-lifted-multicut}(\mu, \lambda)$

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 $\mu_0 := \mu$ 
 $(\delta, \lambda_0) := \text{update-labeling}(\mu_0, \lambda)$ 
let  $\alpha_0 : \mathbb{N} \rightarrow \{0, 1\}$  such that  $\alpha_0(\mathbb{N}) = \{1\}$ 
 $t := 0$ 
repeat
   $\Delta := 0$ 
   $\mu_{t+1} := \mu_t$ 
   $\lambda_{t+1} := \lambda_t$ 
  let  $\alpha_{t+1} : \mathbb{N} \rightarrow \{0, 1\}$  such that  $\alpha_{t+1}(\mathbb{N}) = \{0\}$ 
  for each  $\{m, m'\} \in \binom{\mu(V)}{2}$ 
    if  $\alpha_t(m) = 0 \wedge \alpha_t(m') = 0$ 
      continue
     $(\delta, \mu_{t+1}, \lambda_{t+1}) := \text{update-2-cut}(\mu_{t+1}, \lambda_{t+1}, m, m')$ 
    if  $\delta < 0$ 
       $\alpha_{t+1}(m) := 1$ 
       $\alpha_{t+1}(m') := 1$ 
       $\Delta := \Delta + \delta$ 
     $(\delta, \lambda_{t+1}) := \text{update-labeling}(\mu_{t+1}, \lambda_{t+1})$ 
     $\Delta := \Delta + \delta$ 
  if  $y^{\mu_{t+1}} \notin Y_{GG'}$ 
     $\mu_{t+1} := R(\mu_{t+1})$  (repair heuristic)
     $\Delta := \varphi(x^{\lambda_{t+1}}, y^{\mu_{t+1}}) - \varphi(x^{\lambda_0}, y^{\mu_0})$ 
   $t := t + 1$ 
while  $\Delta < 0$ 

```

Function 3: $(\Delta', \mu', \lambda') = \text{update-2-cut}(\mu, \lambda, m, m')$

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 $\mu_0 := \mu$ 
 $\lambda_0 := \lambda$ 
if  $\mu^{-1}(m') = \emptyset$ 
   $V_0 := \mu^{-1}(m)$ 
else
   $V_0 := \{v \in \mu^{-1}(m) \mid \exists w \in \mu^{-1}(m') : \{v, w\} \in E\}$ 
if  $\mu^{-1}(m) = \emptyset$ 
   $W_0 := \mu^{-1}(m')$ 
else
   $W_0 := \{w \in \mu^{-1}(m') \mid \exists v \in \mu^{-1}(m) : \{v, w\} \in E\}$ 
let  $\alpha : \mathbb{N} \rightarrow \{0, 1\}$  such that  $\alpha(\mathbb{N}) = 1$ 
 $t := 0$ 
while  $V_t \cup W_t \neq \emptyset$ 
   $\delta := \delta' := \infty$ 
  if  $V_t \neq \emptyset$ 
    choose  $(\hat{v}, \hat{l}) \in \underset{(v,l) \in V_t \times L}{\text{argmin}} \varphi(x^{T_{vl}(\lambda_t)}, y^{T'_{vm'}(\mu_t)}) - \varphi(x^{\lambda_t}, y^{\mu_t})$ 
     $\delta := \varphi(x^{T_{\hat{v}\hat{l}}(\lambda_t)}, y^{T'_{\hat{v}m'}(\mu_t)}) - \varphi(x^{\lambda_t}, y^{\mu_t})$ 
  if  $W_t \neq \emptyset$ 
    choose  $(\hat{w}, \hat{l}) \in \underset{(w,l) \in W_t \times L}{\text{argmin}} \varphi(x^{T_{wl}(\lambda_t)}, y^{T'_{wm}(\mu_t)}) - \varphi(x^{\lambda_t}, y^{\mu_t})$ 
     $\delta' := \varphi(x^{T_{\hat{w}\hat{l}}(\lambda_t)}, y^{T'_{\hat{w}m}(\mu_t)}) - \varphi(x^{\lambda_t}, y^{\mu_t})$ 
  if  $\delta \leq \delta'$ 
     $\mu_{t+1} := T'_{\hat{v}m'}(\mu_t)$  (move node  $\hat{v}$  to component  $m'$ )
     $\lambda_{t+1} := T_{\hat{v}\hat{l}}(\lambda_t)$  (label node  $\hat{v}$  with label  $\hat{l}$ )
     $\alpha(\hat{v}) := 0$  (mark  $\hat{v}$  as inactive)
  else
     $\mu_{t+1} := T'_{\hat{w}m}(\mu_t)$  (move node  $\hat{w}$  to component  $m$ )
     $\lambda_{t+1} := T_{\hat{w}\hat{l}}(\lambda_t)$  (label node  $\hat{w}$  with label  $\hat{l}$ )
     $\alpha(\hat{w}) := 0$  (mark  $\hat{w}$  as inactive)
   $V_{t+1} := \{v \in V \mid \mu_{t+1}(v) = m \wedge \alpha(v) = 1 \wedge \exists \{v, w\} \in E : \mu_{t+1}(w) = m'\}$ 
   $W_{t+1} := \{w \in V \mid \mu_{t+1}(w) = m' \wedge \alpha(w) = 1 \wedge \exists \{v, w\} \in E : \mu_{t+1}(v) = m\}$ 
   $t := t + 1$ 
   $\hat{t} := \min_{t' \in \{0, \dots, t\}} \underset{t' \in \{0, \dots, t\}}{\text{argmin}} \varphi(x^{\lambda_{t'}}, y^{\mu_{t'}}) - \varphi(x^{\lambda_0}, y^{\mu_0})$ 
   $\Delta_1 := \varphi(x^{\lambda_{\hat{t}}}, y^{\mu_{\hat{t}}}) - \varphi(x^{\lambda_0}, y^{\mu_0})$ 
   $\Delta_2 := \varphi(x^{\lambda_0}, y^{T'_{mm'}(\mu)}) - \varphi(x^{\lambda_0}, y^{\mu_0})$  (join  $m$  and  $m'$ )
  if  $\min\{\Delta_1, \Delta_2\} \geq 0$ 
    return  $(0, \mu, \lambda)$ 
  else if  $\Delta_1 < \Delta_2$ 
    return  $(\Delta_1, \mu_{\hat{t}}, \lambda_{\hat{t}})$ 
  else
    return  $(\Delta_2, T_{mm'}(\mu), \lambda)$ 

```

Application-specific accuracy. In Tab. 1, we compare feasible solutions output by KLj-r and KLj+r after convergence with those obtained by B&C after at most three hours. It can be seen from this table that the feasible solutions output by KLj-r and KLj+r have lower cost

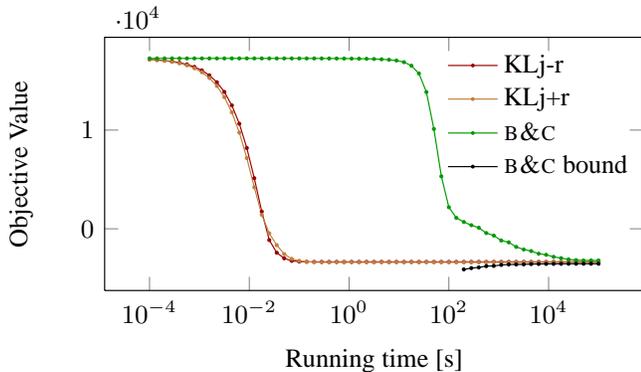


Figure 3: Convergence of B&C, KLj-r and KLj+r in an application to the task of articulated human body pose estimation.

$ V $	Algorithm	Acc [%]	Mean obj.	Mean t [s]	Median t [s]
150	B&C [9]	56.53	-3013.30	9519.26	308.28
	KLj-r	58.20	-3352.74	0.033	0.031
	KLj+r	57.55	-3419.07	0.119	0.100
420	KLj-r	60.85	-6184.36	0.098	0.053
	KLj+r	60.58	-6608.53	0.534	0.254

Table 1: Comparison of B&C, KLj-r and KLj+r in an application to the task of articulated human body pose estimation.

and higher application-specific accuracy (Acc) on average. KLj+r yields a lower average cost than KLj-r with slightly higher running time. The fact that lower cost does not mean higher application-specific accuracy is explained by the application-specific accuracy measure that does not penalize false positives.

The shorter absolute running time of KLj-r and KL+r allows us to increase the number of nodes from 150, as in [9], to 450. It can be seen from the last two rows of Tab. 1 that this increases the application-specific accuracy by about 2.5%.

4.2. Multiple Object Tracking

We turn toward applications of the NL-LMP and the algorithms KLj-r and KLj+r to the task of multiple object tracking. Tang et al. [26] approach this problem via a graph decomposition and node labeling problem that we identify as a special case of the NL-LMP with two labels and subgraph selection (Sec. 2.5.3). We relate their notation to ours rigorously in the supplement of this paper. Nodes in their graph are putative detections of persons. In our notation, $x_{vl} = 1$ indicates that the putative detection v is active, and $y_{vw} = 1$ indicates that the putative detections v and w are interpreted in the solution as detections of distinct persons. For the test

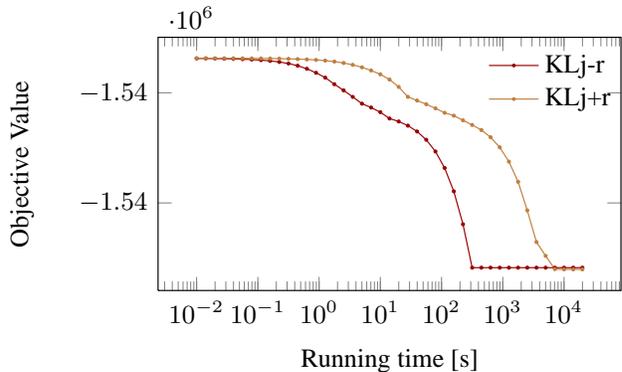


Figure 4: Convergence of the algorithms KLj-r and KLj+r in an application to the task of multiple object tracking.

set of the multiple object tracking benchmark [16], Tang et al. construct seven such instances of the NL-LMP.

To tackle these instances, Tang et al. solve the subgraph suppression problem first and independently, by thresholding, and then solve the minimum cost multicut problem for the remaining subgraph by means of the algorithm KLj of [12], without re-iterating. Here, we apply to the joint NL-LMP the algorithms KLj-r and KLj+r and compare their output to [26] and to other top-performing algorithms [8, 13, 4, 27].

Cost and time. The convergence of the algorithms KLj-r and KLj+r is shown in Fig. 4. It can be seen from this figure that KLj-r converges faster than KLj+r.

Application-specific accuracy. We compare the feasible solutions output by KLj-r and KLj+r to the state-of-the-art for the benchmark [16]. To this end, we report in Tab. 2 the standard CLEAR MOT metric, including: multiple object tracking accuracy (MOTA), multiple object tracking precision (MOTP), mostly tracked object (MT), mostly lost (ML) and tracking fragmentation (FM). MOTA combines identity switches (ID Sw), false positives (FP) and false negatives (FN) and is most widely used. Our feasible solutions are published also at the benchmark website under the names NLLMP (KLj-r) and NLLMPj (KLj+r). It can be seen from Tab. 2 that the feasible solutions obtained by KLj-r and KLj+r rank first in MOTA and MOTP. Compared to [26], KLj-r and KLj+r reduce the number of false positives and false negatives. The average inverse running time per frame of a video sequence (column “Hz” in the table) is better for KLj-r by a margin than for any other algorithm. Overall, these results show the practicality of the NL-LMP in conjunction with the local search algorithms KLj-r and KLj+r for applications in multiple object tracking.

Method	MOTA \uparrow	MOTP \uparrow	FAF \downarrow	MT \uparrow	ML \downarrow	FP \downarrow	FN \downarrow	ID Sw \downarrow	Frag \downarrow	Hz \uparrow	Detector
[8]	40.1	74.8	1.3	11.6%	51.3%	7896	99224	430	963	1.1	Public
[13]	42.8	76.6	1.0	13.6%	46.9%	5668	97919	499	659	0.8	Public
[4]	46.4	76.6	1.6	18.3%	41.4%	9753	87565	359	504	2.6	Public
[27]	46.3	75.7	1.1	15.5%	39.7%	6449	90713	663	1115	0.8	Public
KLj-r	47.6	78.5	1.0	17.0%	40.4%	5844	89093	629	768	8.3	Public
KLj+r	47.6	78.5	0.98	17.0%	40.4%	5783	89160	627	761	0.7	Public

Table 2: Comparison of the algorithms KLj-r and KLj+r in an application to the task of multiple object tracking.

5. Conclusion

We have stated the minimum cost node labeling lifted multicut problem, NL-LMP, an NP-hard combinatorial optimization problem whose feasible solutions define both a decomposition and a node labeling of a given graph. We have defined and implemented two local search algorithms, KLj-r and KLj+r, that converge monotonously to a local optimum, offering a feasible solution at any time. We have shown applications of these algorithms to the task of articulated human body pose estimation and to the task of multiple object tracking, obtaining competitive results. We conclude that the NL-LMP is a useful mathematical abstraction in the field of computer vision and is practical, despite its NP-hardness, in conjunction with local search algorithms.

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