A Unifying Theory for Central Panoramic Systems and Practical Implications

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Abstract. Omnidirectional vision systems can provide panoramic alertness in surveillance, improve navigational capabilities, and produce panoramic images for multimedia. Catadioptric realizations of omnidirectional vision combine reflective surfaces and lenses. A particular class of them, the central panoramic systems, preserve the uniqueness of the projection viewpoint. In fact, every central projection system including the well known perspective projection on a plane falls into this category.

In this paper, we provide a unifying theory for all central catadioptric systems. We show that all of them are isomorphic to projective mappings from the sphere to a plane with a projection center on the perpendicular to the plane. Subcases are the stereographic projection equivalent to parabolic projection and the central planar projection equivalent to every conventional camera. We define a duality among projections of points and lines as well as among different mappings.

This unification is novel and has a a significant impact on the 3D interpretation of images. We present new invariances inherent in parabolic projections and a unifying calibration scheme from one view. We describe the implied advantages of catadioptric systems and explain why images arising in central catadioptric systems contain more information than images from conventional cameras. One example is that intrinsic calibration from a single view is possible for parabolic catadioptric systems given only three lines. Another example is metric rectification using only affine information about the scene.

1 Introduction

Artificial visual systems face extreme difficulties in tasks like navigating on uneven terrain or detecting other movements when they are moving themselves. Paradoxically, these are tasks which biological systems like insects with very simple brains can very easily accomplish. It seems that this is not a matter of computational power but a question of sensor design and representation. The representation of visual information has to be supported by the adequate sensors in order to be direct and efficient. It is therefore surprising that most artificial visual systems use only one kind of sensor: a CCD-camera with a lens.

We believe that the time has come to study the question of representation in parallel to the design of supportive sensing hardware. As in nature these sensors and representations should depend on the tasks and the physiology of the

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observer. Omnidirectional or panoramic visual sensors are camera designs that enable capturing of a scene with an almost hemi-spherical field of view. Originally introduced mainly for monitoring activities they now are widely used in multimedia and robotics applications. The advantages of omnidirectional sensing are obvious for applications like surveillance, immersive telepresence, videoconferencing, mosaicing, and map building. A panoramic field of view eliminates the need for more cameras or a mechanically turnable camera. We prove in this paper that a class of omnidirectional sensors, the central panoramic systems, can recover information about the environment that conventional models of perspective projection on a plane cannot.

First let us summarize recent activities on omnidirectional vision. A panoramic field of view camera was first proposed by Rees [13]. After 20 years the concept of omnidirectional sensing was reintroduced in robotics [16] for the purpose of autonomous vehicle navigation. In the last five years, several omnidirectional cameras have been designed for a variety of purposes. The rapid growth of multimedia applications has been a fruitful testbed for panoramic sensors [7,8,11] applied for visualization. Another application is telepresence [14,1] where the panoramic sensor achieves the same performance as a remotely controlled rotating camera with the additional advantage of an omnidirectional alert awareness. Srinivasan [2] designed omnidirectional mirrors that preserve ratios of elevations of objects in the scene and Hicks [5] constructed a mirror-system that rectifies planes perpendicular to the optical axis. The application of mirror-lens systems in stereo and structure from motion has been prototypically described in [15,4]. Our work is hardly related to any of the above approaches. The fact that lines project to conics is mentioned in the context of epipolar lines by Svoboda [15] and Nayar [10].

Omnidirectional sensing can be realized with dioptric or catadioptric systems. Dioptric systems consist of fish-eye lenses while catadioptric systems are combinations of mirrors and lenses. These sensors can be separated into two classifications, determined by whether they have a unique effective viewpoint. Conical and spherical mirror systems as well as most fish-eye lenses do not possess a single vantage-point. Among those that do have a unique effective viewpoint are systems which are composed of multiple planar mirrors and perspective cameras all of whose viewpoints coincide, as well as a hyperbolic mirror in front of a perspective camera, and a parabolic mirror in front of a orthographic camera. The uniqueness of a projection point is equivalent to a purely rotating planar camera with the nice property that a rotated image is a collineation of an original one. Hence, every part of an image arising from such a catadioptric sensor can easily re-warped into the equivalent image of a planar camera looking to the desired direction without knowledge of the depths of the scene. It is worth mentioning that simple dioptric systems — conventional cameras — are included in this class of catadioptric systems because they are equivalent to catadioptric systems with a planar mirror.

In this paper, we present a unifying theory for all central panoramic systems, that means for all catadioptric systems with a unique effective viewpoint. We prove that all cases of a mirror surface—parabolic, hyperbolic, elliptic, planar—

with the appropriate lens—orthographic or perspective—can be modeled with a projection from the sphere to the plane where the projection center is on a sphere diameter and the plane perpendicular to it. Singular cases of this model are stereographic projection, which we show to be equivalent to the projection induced by a parabolic mirror through a orthographic lens, and central projection which is well known to be equivalent to perspective projection.

Given this unifying projection model we establish two kinds of duality: a duality among point projections and line projections and a duality among two sphere projections from two different centers. We show that parallel lines in space are projected onto conics whose locus of foci is also a conic. This conic is the horizon of the plane perpendicular to all of the original lines, but the horizon is obtained via the dual projection. In case of perspective projection all conics degenerate to lines and we have the well known projective duality between lines and points in P^2.

The practical implications are extremely useful. The constraints given by the projection of lines are natural for calibration by lines. We prove that three lines are sufficient for intrinsic calibration of the catadioptric system without any metric information about the environment. We give a natural proof why such a calibration is not possible for conventional cameras showing thus the superiority of central catadioptric systems. The unifying model we have provided allows us to study invariances of the projection. Perhaps most importantly, in the case of parabolic systems we prove that angles are preserved because the equivalent projection—stereographic—is a conformal mapping. This allows us to estimate the relative position of the plane and facilitates a metric rectification of a plane without any assumption about the environment.

In section 2 we prove the equivalence of the catadioptric and spherical projections and develop the duality relationships. In section 3 we present the computational advantages derived from this theory and in section 4 we show our experimental results.

2 Theory of Catadioptric Image Geometry

The main purpose of this section is to prove the equivalence of the image geometries obtained by the catadioptric projection and the composition of projections of a sphere. We first develop the general spherical projection, and then the catadioptric projections, showing in turn that each are equivalent to some spherical projection. Then we will show that two projections of the sphere are dual to each other, and that parabolic projection is dual to perspective projection.

2.1 Projection of the Sphere to a Plane

We introduce here a map from projective space to the sphere to an image plane. A point in projective space is first projected to an antipodal point pair on the sphere. An axis of the sphere is chosen, as well as a point on this axis, but within the sphere. From this point the antipodal point pair is projected to a pair of points on a plane perpendicular to the chosen axis.

Assume that the sphere is the unit sphere centered at the origin, the axis is the z-axis and the point of projection is the point $(0, 0, l)$. Let the plane $z = -m$ be the image plane. If $\sim$ is the equivalence relation relating antipodal points on the sphere, then the map from projective space to the sphere $s : P^3(\mathbb{R}) \to S^2/\!\sim$ is given by

$$s(x, y, z, w) = \left(\pm\frac{x}{r}, \pm\frac{y}{r}, \pm\frac{z}{r}\right)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. To determine the second part of the map, we need only determine the perspective projection to the plane $z = -m$ from the point $(0, 0, l)$. Without taking the equivalence relation into account the projection of (x, y, z) is

$$p_{l,m}(x, y, z) = \left(\frac{x(l+m)}{lr - z}, \frac{y(l+m)}{lr - z}, -m\right) \ .$$

Now applying the equivalence relation we have a map $p^*_{l,m} : P^3(\mathbb{R}) \to \mathbb{R}^2_\sim$,

$$p^*_{l,m}(x, y, z, w) = \left(\pm\frac{x(l+m)}{lr \mp z}, \pm\frac{y(l+m)}{lr \mp z}, -m\right) \ .$$

Here $\mathbb{R}^2_\sim$ is $\mathbb{R}^2$ with the equivalence relation induced on it by the map $p_{l,m}$ and $\sim$ on the sphere.

If we move the projection plane to $z = -\alpha$, then the relation between the two projections is

$$p^*_{l,m}(x, y, z, w) = \frac{l+m}{l+\alpha} p^*_{l,\alpha}(x, y, z, w) \ .$$

So they are the same except for a scale factor. Thus if m is not indicated it is assumed that $m = 1$.

Remark. When $l = 1$ and $m = 0$, i.e. the point of projection is the north pole, we obtain

$$p^*_{1,0}(x, y, z, w) = \left(\pm\frac{x}{\sqrt{x^2 + y^2 + z^2} \mp z}, \pm\frac{y}{\sqrt{x^2 + y^2 + z^2} \mp z}\right) \ ,$$

which is a case of stereographic projection [9] (when (x, y, z) is restricted to the sphere). On the other hand, when $l = 0$ and $m = 1$, we have perspective projection:

$$p^*_{0,1}(x, y, z, w) = \left(\frac{x}{z}, \frac{y}{z}\right) \ .$$

2.2 Catadioptric Projection

In this section we will describe the projections using conical section mirrors. Throughout the section we will refer to figures 2 and 3.

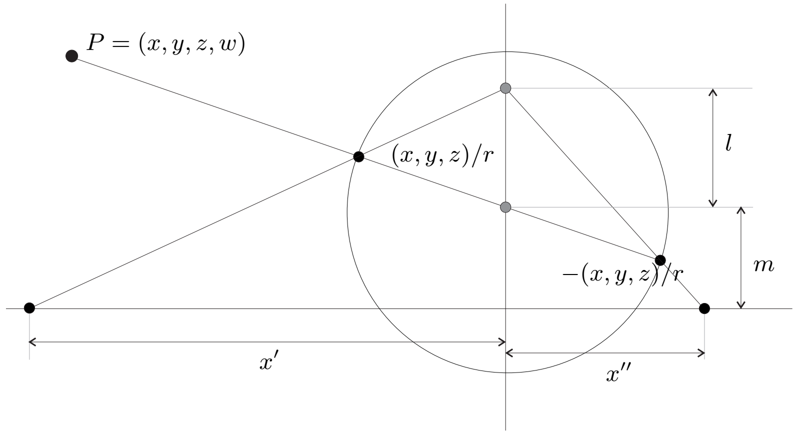

Fig. 1. A point $P = (x, y, z, w)$ is projected via s to two antipodal points $(\pm x, \pm y, \pm z)/r$ on the sphere. The two antipodal points are projected to the image plane $z = -m$ via projection from the point $(0, 0, l)$.

Parabolic Mirror. We call the projection induced by a parabolic mirror to an image plane a parabolic projection. The parabolic projection of a point P in space is the orthographic projection of the intersection of the line PF (where F is the parabola's focus) and the parabola. The orthographic projection is to the image plane perpendicular to the axis of the parabola. Any line (in particular a ray of light) incident with the focus is reflected such that it is perpendicular to the image plane, and (ideally) these are the only rays that the orthographic camera receives.

The projection described is equivalent to central projection of a point to the parabola, followed by standard orthographic projection. Thus we proceed in a similar fashion as we did for the sphere. Assume that a parabola is placed such that its axis is the z-axis, its focus is located at the origin, and p is its focal length. Then

$$S = \left\{ (x, y, z) \,\Big|\, \frac{1}{4p}(x^2 + y^2 - p = z \right\}$$

is the surface of the parabola. Now define $\sim$ such that if $P, Q \in S$, then $P \sim Q$ if and only if there exists a $\lambda \in \mathbb{R}$ such that $P = \lambda Q$. We now determine the projection $s_p : P^3(\mathbb{R}) \to S/\!\sim$,

$$s_p(x, y, z, w) = \left(\pm\frac{2px}{r \mp z}, \pm\frac{2py}{r \mp z}, \pm\frac{2pz}{r \mp z} \right) ,$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The next step is to project orthographically to the plane $z = 0$ (the actual distance of the plane to the origin is of course inconsequential). We thereby obtain $q_p^* : P^3(\mathbb{R}) \to \mathbb{R}^2/\!\sim$ given by

$$q_p^*(x, y, z, w) = \left(\pm\frac{2px}{r \mp z}, \pm\frac{2py}{r \mp z} \right) .$$

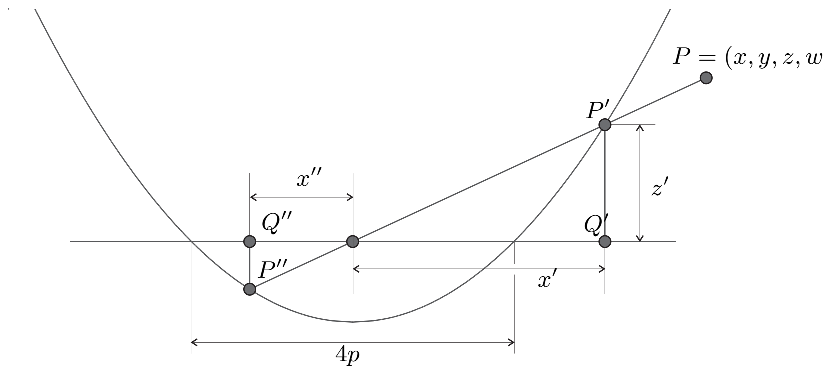

Fig. 2. Cross-section of a parabolic mirror. The image plane is through the focal point. The point in space P is projected to the antipodal points P' and P'', which are then *orthographically* projected to Q' and Q'' respectively.

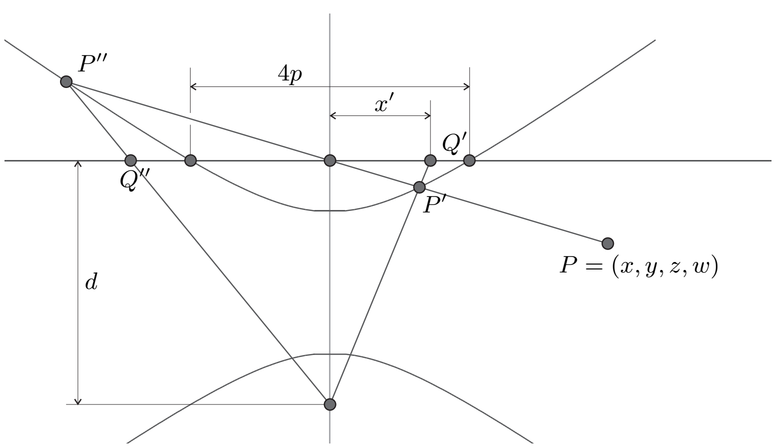

Fig. 3. Cross-section of a hyperbolic mirror, again the image plane is through the focal point. The point in space P is projected to the antipodal points P' and P'', which are then *perspectively* projected to Q' and Q'' from the second focal point.

Again, $\mathbb{R}^2/\!\sim$ is $\mathbb{R}^2$ with the equivalence relation carried over by orthographic projection of the parabola.

Remark. Note that

$$q_p^*(x,y,z,w) = 2p\, p_{1,0}^*(x,y,z,w) = p_{1,2p-1}^*(x,y,z,w) \ .$$

Hyperbolic and Elliptical Mirrors. As with the paraboloid, hyperbolic projection is the result of reflecting rays off of a hyperbolic mirror. Rays incident with one the focal points of the hyperbola are reflected into rays incident with the second focal point. To obtain the projection of a point, intersect the line containing the point and the focal point with the hyperbola. Take the two intersection points and projection them to the image plane. The same applies to ellipses.

Assume a hyperbola is placed such that its axis is the z-axis, one of its foci is the origin, the other $(0,0,-d)$, and its latus rectum is $4p$. Then the surface of the hyperbola is

$$S = \left\{ (x,y,z) \,\middle|\, \left(\frac{z+d/2}{a}\right)^2 - \left(\frac{x}{b}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \right\}$$

where

$$a = \frac{1}{2}\left(\sqrt{d^2+4p^2} - 2p\right), \text{ and } b = \sqrt{p\sqrt{d^2+4p^2} - 2p^2} \ .$$

Let $\sim$ be similarly defined for points of S, identifying antipodal points of the hyperbola's surface with respect to the focus. The projection $s_{p,d}(x,y,z,w) : P^3(\mathbb{R}) \to S/\!\sim$ may be obtained by intersecting the line through the point and the origin, however it is of too great a length to include here. Nevertheless, once obtained we then proceed by applying a perspective projection of the the antipodal point pair given by $s_{p,d}(x,y,z,w)$ from the point $(0,0,-d)$ to the plane $z=0$, calling this projection $r_{p,d}^* : P^3(\mathbb{R}) \to \mathbb{R}^2/\!\sim$. We find that

$$r_{p,d}^*(x,y,z,w) = \left(\pm \frac{2xdp/\sqrt{d^2+4p^2}}{\frac{d}{\sqrt{d^2+4p^2}} r \mp z}, \pm \frac{2ydp/\sqrt{d^2+4p^2}}{\frac{d}{\sqrt{d^2+4p^2}} r \mp z} \right) \ ,$$

where $r = \sqrt{x^2+y^2+z^2}$.

Remark. Notice that

$$r_{p,d}^*(x,y,z,w) = p^*_{\frac{d}{\sqrt{d^2+4p^2}}, \frac{d(1-2p)}{\sqrt{d^2+4p^2}}}(x,y,z,w) \ .$$

For an ellipsoid similarly placed so that its foci are $(0,0,0)$ and $(0,0,-d)$, and latus rectum of $4p$, we have

$$S = \left\{ (x,y,z) \,\middle|\, \left(\frac{z+d/2}{a}\right)^2 + \left(\frac{x}{b}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \right\} \ ,$$

where

$$a = \frac{1}{2}\left(\sqrt{d^2+4p^2}+2p\right), \text{ and } b = \sqrt{p\sqrt{d^2+4p^2}+2p^2} \ .$$

We then derive that $t^*_{p,d}(x,y,z,w) : P^3(\mathbb{R}) \to \mathbb{R}^2/\sim$ is given by

$$t^*_{p,d}(x,y,z,w) = \left(\pm\frac{2xdp}{dr \pm z\sqrt{d^2+4p^2}}, \pm\frac{2ydp}{dr \pm z\sqrt{d^2+4p^2}}\right) \ .$$

Remark. We have that $t^*_{p,d}$ satisfies

$$t^*_{p,d}(x,y,z,w) = r^*_{p,d}(x,y,-z,w) = p^*_{\frac{d}{\sqrt{d^2+4p^2}},\frac{d(1-2p)}{\sqrt{d^2+4p^2}}}(x,y,-z,w) \ .$$

Thus the ellipse gives the same projection as the hyperbola, modulo a reflection about $z = 0$.

2.3 Equivalence of Catadioptric and Spherical Projections

From the discussion above we may write a general theorem which will allow us to more generally develop the theory of catadioptric image geometry. We have the following central theorem.

Theorem 1 (Projective Equivalence). *Catadioptric projection with a single effective viewpoint is equivalent to projection to a sphere followed by projection to a plane from a point.*

Proof. In the past two sections we have the following relationships for the projection functions:

$$\begin{aligned}
r^*_{p,d}(x,y,z,w) &= p^*_{\frac{d}{\sqrt{d^2+4p^2}},\frac{d(1-2p)}{\sqrt{d^2+4p^2}}}(x,y,z,w) && \text{(hyperbola} \longleftrightarrow \text{sphere)} \ , \\
t^*_{p,d}(x,y,z,w) &= p^*_{\frac{d}{\sqrt{d^2+4p^2}},\frac{d(1-2p)}{\sqrt{d^2+4p^2}}}(x,y,-z,w) && \text{(ellipse} \longleftrightarrow \text{sphere)} \ , \\
q^*_p(x,y,z,w) &= p^*_{1,2p-1}(x,y,z,w) && \text{(parabola} \longleftrightarrow \text{sphere)} \ , \\
\left(\tfrac{fx}{z}, \tfrac{fy}{z}\right) &= p^*_{0,f}(x,y,z,w) && \text{(perspective} \longleftrightarrow \text{sphere)} \ .
\end{aligned}$$

Each are maps from $P^3(\mathbb{R})$ to $\mathbb{R}^2_\sim$, and for any point in space the relations show that they map to the same point in the image plane. □

We now have a unified theory of catadioptric projection, and in further discussion we need only consider projections of the sphere. In the interest of conciseness we wish to give a name to this class of projections. We write $\pi_{l,m}$ to represent the projective plane induced by the projection $p^*_{l,m}$. Recall that if $l = 1$ then we have the projective plane obtained from stereographic projection, or equivalently parabolic projection. If $l = 0$ then we have the projective plane obtained from perspective projection.

Having demonstrated the equivalence with the sphere we now wish to describe in more detail the structure of the projective plane $\pi_{l,m}$. We therefore describe the images of lines under these projections, therefore the "lines" of the projective planes. But because of the equivalence with the sphere, we may restrict ourselves to studying the projections of great circles and antipodal points. Thus let $s_{l,m} : S^2/\sim \rightarrow \mathbb{R}^2/\sim$ project points of the sphere to the image plane. Figure 4 shows the projection of the great circle to the image plane, the equator is projected to a circle of radius $\frac{l+m}{l}$; this is the horizon of the fronto-parallel since the equator is the projection of the line at infinity in the plane $z = 0$. The proposition below describes the family of conics which are images of lines.

Proposition 1. *The image of a line is a conic whose major axis (when it exists) intersects the image center. It has the property that it intersects the fronto-parallel horizon antipodally and its major axis intersects the image center.*

Proof. Note first that the intersection of a great circle (which is itself the image of a line in space) with the equator are two points which are antipodal. Their projection to the image plane gives two points which again are antipodal on the image of the equator. The image of the great circle must be symmetric about the axis made by the perpendicular bisector of the two intersection points. This axis contains the image center since the midpoint of the intersection points is the image center.

The actual image may be obtained by taking a cone whose vertex is the point of projection $(0, 0, l)$ and which contains the great circle, then intersecting the cone with the image plane. The intersection of a plane and a skew cone is still a conic. □

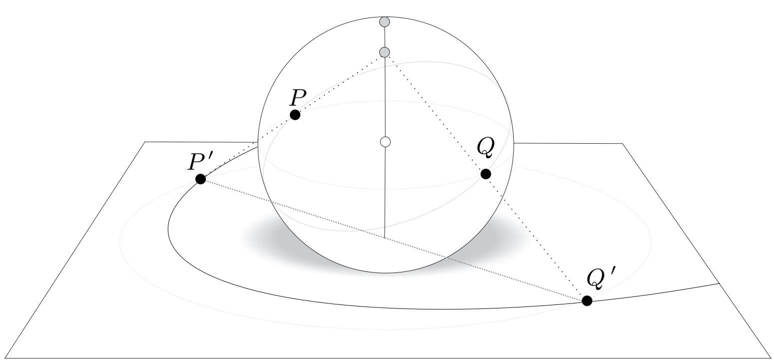

Fig. 4. A line in space is projected to a great circle on the sphere, which is then projected to a conic section in the plane via $p^*_{l,m}$. The equator is mapped to the fronto-parallel horizon, the dotted circle in the plane.

Note that if a conic has the properties in the proposition it is not necessarily the image of a line. There is an additional constraint on the foci of the conic. Let us therefore determine the image of a great circle. Let $\hat{n} = (n_x, n_y, n_z)$ be the normal of a plane containing some great circle. To obtain the quadratic form of the conic section we find the quadratic form of the cone through $(0,0,l)$ and the great circle. To do this we first rotate to a coordinate system (x',y',z') such that the great circle lies in the plane $z'=0$. Then the vertex of the cone is $(l\sqrt{1-n_z^2},0,ln_z)$. Points of the cone, in the rotated coordinate system, then satisfy

$$p'\begin{pmatrix} 1 & 0 & \frac{\sqrt{1-n_z^2}}{n_z} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{1-n_z^2}}{n_z} & 0 & \frac{1-\frac{1}{l^2}-n_z^2}{n_z^2} & \\ 0 & 0 & \frac{1}{ln_z} & -1 \end{pmatrix} p'^T \ .$$

By rotating back to the original coordinate system we have,

$$p\begin{pmatrix} -n_x^2\alpha - l^2(n_x^2+n_y^2n_z^2) & (l^2-1)n_xn_y\alpha & ln_xn_z\alpha \\ (l^2-1)n_xn_y\alpha & -n_y^2\alpha - l^2(n_y^2+n_x^2n_z^2) & ln_yn_z\alpha \\ ln_xn_z\alpha & ln_yn_z\alpha & -l^2n_z^2\alpha \end{pmatrix} p^T \ ,$$

where $\alpha = n_z^2 - 1 = n_x^2 + n_y^2$. Let $C_{\hat{n}}$ be the matrix above. From this form we may extract the axes, center, eccentricity and foci, finding that

$$\begin{aligned} c &= \left(\tfrac{(l+m)n_x|n_z|}{n_x^2+n_y^2-l^2}, \tfrac{(l+m)n_y|n_z|}{n_x^2+n_y^2-l^2}\right) && \text{(center)}\ , \\ f_\pm &= \left(\tfrac{(l+m)n_x\left(|n_z|\pm\sqrt{1-l^2}\right)}{n_x^2+n_y^2-l^2}, \tfrac{(l+m)n_y\left(|n_z|\pm\sqrt{1-l^2}\right)}{n_x^2+n_y^2-l^2}\right) && \text{(foci)}\ , \\ a &= \tfrac{l(l+m)n_z}{l^2-n_x^2-n_y^2} && \text{(minor axis)}\ , \\ b &= \tfrac{l+m}{\sqrt{l^2-n_x^2-n_y^2}} && \text{(major axis)}\ , \\ \epsilon &= \sqrt{\tfrac{(1-l^2)(n_x^2+n_y^2)}{l^2-n_x^2-n_y^2}} && \text{(eccentricity)}\ . \end{aligned}$$

Meet and join. We find that the set of "points" of the projective plane $\pi_{l,m}$,

$$\Pi(\pi_{l,m}) = \left\{ \left(\pm\frac{(l+m)m_x}{l\mp m_z}, \pm\frac{(l+m)m_y}{l\mp m_z} \right) \middle| \hat{n}\in S^2 \right\} \ .$$

A line is the set of points,

$$[\hat{n}] = \left\{ (x,y) \,\middle|\, (x\ y)\, C_{\hat{n}} \begin{pmatrix} x \\ y \end{pmatrix} \right\} \ .$$

Thus the set of "lines" of the projective plane $\pi_{l,m}$,

$$\Lambda(\pi_{l,m}) = \left\{ [\hat{n}] \,\middle|\, \hat{n}\in S^2 \right\} \ .$$

We may then define the operator meet $\wedge : \Lambda(\pi_{l,m})\times\Lambda(\pi_{l,m}) \to \Pi(\pi_{l,m})$ to take a pair of lines to their intersection, and the operator join $\vee : \Pi(\pi_{l,m})\times\Pi(\pi_{l,m}) \to \Lambda(\pi_{l,m})$ to take a pair of points to the line through them.

2.4 Duality

In this section we will show that two projections of the sphere are dual to each other. The antipodal point pairs of one projection are the foci of line images in another projection, and vice versa. When the projection is stereographic (i.e. parabolic) the dual is the usual perspective projection.

We have seen that images of lines are conics, we would like to know if there is anything special about families of line images which intersect the same point. A set of longitudes on the sphere all intersect in two antipodal points, what are their projections? It is clear that the images must all intersect in two points since incidence relationships are preserved, but is there anything special about this particular pencil of conics?

Proposition 2. *The locus of foci of a set of line images, where the great circles corresponding to the lines intersect antipodally, is a conic whose foci are the images of the intersection points.*

Proof. Assume l and m are constant. Choose some point $\hat{m} = (m_x, m_y, m_z)$ on the sphere, by rotational symmetry we may assume without loss of generality that $m_y = 0$. The normals of all lines perpendicular to $\hat{m}$, i.e. those which intersect $\hat{m}$, are

$$(n_x^\theta, n_y^\theta, n_z^\theta) = (m_x \sin\theta, \cos\theta, m_z \sin\theta) \ .$$

Substituting into the formula found for the first focus, we have

$$\begin{aligned} f_1^\theta &= \left(\frac{(l+m)n_x^\theta \left(n_z^\theta + \sqrt{1-l^2}\right)}{{n_x^\theta}^2 + {n_y^\theta}^2 - l^2}, \frac{(l+m)n_y^\theta \left(n_z^\theta + \sqrt{1-l^2}\right)}{{n_x^\theta}^2 + {n_y^\theta}^2 - l^2} \right) \\ &= \left(\frac{(l+m)m_x \sin\theta}{\sqrt{1-l^2} - m_z \sin\theta}, \frac{(l+m)\cos\theta}{\sqrt{1-l^2} - m_z \sin\theta} \right) \ . \end{aligned}$$

But this is just the projection of $(n_x^\theta, n_y^\theta, n_z^\theta)$ by

$$p^*_{\sqrt{1-l^2}, l+m-\sqrt{1-l^2}} \ .$$

So let $l' = \sqrt{1-l^2}$ and $m' = l + m - \sqrt{1-l^2}$. Under the projection $p^*_{l',m'}$ the image of the great circle perpendicular to (m_x, m_y, m_z), i.e. the points $\{(n_x^\theta, n_y^\theta, n_z^\theta)\}$, is once again a conic. One of its foci is

$$f_1' = \left(\frac{(l'+m')m_x \left(m_z + \sqrt{1-l'^2}\right)}{n_x^2 - l'^2}, 0 \right) = \left(\frac{(l+m)m_x}{l - m_z}, 0 \right) \ .$$

This is the image of $(m_x, 0, m_y)$ under $p^*_{l,m}$. □

Define the map $f_{l,m}$ such that given the normal of a line it produces the foci of the line's image. Note that this map is injective and therefore its inverse is well defined. We have the following theorem on duality.

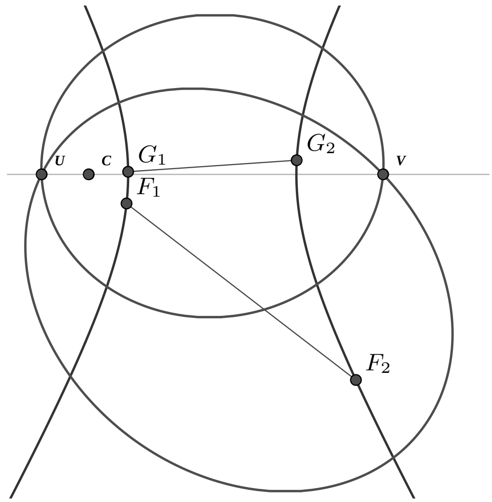

Fig. 5. The two ellipses are projections of two lines in space. Their foci F_1, F_2, and G_1, G_2 respectively lie on a hyperbola containing the foci of all *all* ellipses through U and V. The foci of this hyperbola are the points U and V.

Theorem 2 (Duality). *Given the two projective planes $\pi_1 = \pi_{l,m}$ and $\pi_2 = \pi_{l',m'}$ where l, m, l' and m' satisfy*

$$l^2 + {l'}^2 = 1 \quad \textit{and} \quad l + m = l' + m' \ ,$$

the following is true,

$$\begin{aligned} f_{l,m} &: \Lambda(\pi_1) \to \Pi(\pi_2), \\ f_{l,m}^{-1} &: \Pi(\pi_2) \to \Lambda(\pi_1), \\ f_{l',m'} &: \Lambda(\pi_2) \to \Pi(\pi_1), \\ f_{l',m'}^{-1} &: \Pi(\pi_1) \to \Lambda(\pi_2) \ . \end{aligned}$$

In fact the two projective planes π_1 and π_2 are dual. If l_1, l_2 are lines of π_1 and p_1, p_2 are points of π_1, then:

$$\begin{aligned} f_{l',m'}^{-1}(l_1 \wedge l_2) &= f_{l,m}(l_1) \vee f_{l,m}(l_2) \\ f_{l,m}(p_1 \vee p_2) &= f_{l',m'}^{-1}(p_1) \wedge f_{l',m'}^{-1}(p_2) \ . \end{aligned}$$

Proof. The preceding proposition showed that the foci of a pencil of lines $\{l_\lambda\}$ lied on a conic c, where $c \in \Lambda(\pi_2)$. The foci of c were the two points of intersection of the pencil of lines, so $f_{l',m'}(c) = l_{\lambda_1} \wedge l_{\lambda_2}$. But $c = f_{l,m}(l_{\lambda_1}) \vee f_{l,m}(l_{\lambda_2})$, and so

$$f_{l',m'}^{-1}(l_{\lambda_1} \wedge l_{\lambda_2}) = f_{l,m}(l_{\lambda_1}) \vee f_{l,m}(l_{\lambda_2}) \ .$$

The second is true because so is the dual to the proposition, namely that a set of collinear points (in π_1) produce a line whose foci are a single point of π_2. □

Corollary 1. *The projective planes $\pi_{1,0}$ and $\pi_{0,1}$ are dual. The first is obtained from stereographic projection, and the second from perspective projection. The center of every circle in a parabolic projection is a point in the perspective projection; every point of a parabolic projection is a focal point of a line in a perspective projection.*

3 Advantages of Catadioptric Projection

The presented unifying theory of catadioptric projections enables a direct and natural insight on the invariances of these projections. The perspective projection is a degenerate case of a catadioptric projection which fact as we will show directly reveals its inferiority to the other catadioptric projections (parabolic and hyperbolic).

3.1 Recovery of Geometric Properties

We have shown that parabolic projection is equivalent to stereographic projection, as well as being dual to perspective projection. Stereographic projection is a map with several important properties. First the projection of any circle on the sphere is a circle in the plane. In particular the projection of a great circle is a circle. What is also important is that the map is conformal. The angle between two great circles (i.e. the inverse cosine of the dot product of the normals of their planes) is the same angle between the circles which are their projections. This is important because it means for one thing that if two circles are horizons of some planes, and they are orthogonal, then the planes are perpendicular.

Corollary 2. *The angle between great circles on the sphere is equal to the angle between their projections.*

Proof of this fact is given in almost any book on geometry, e.g. [12], and is a direct result of the fact that stereographic projection is a conformal mapping. This implies that the angles between the horizons of two planes is equal to the angle between the two planes; orthogonal planes have orthogonal horizons.

3.2 Calibration

Almost all applications in computer vision require that the imaging sensor's intrinsic parameters be calibrated. The intrinsic parameters include focal length, image center and aspect ratio, as well as any other parameters which determine the projection induced by the sensor such as radial distortion. Sometimes it is possible to calibrate one or more of those parameters with minimal prior information about scene geometry or configuration. For example, it has been shown that radial distortion can be calibrated for, using only the images of lines. The only assumption is that points have been gathered in the image which are projections of points in space lying on some straight line. Using this information not only is it possible to determine the radial distortion parameters, but the image center also may be obtained.

We have shown prior to this work that it is possible to calibrate all of the intrinsic parameters of a parabolic catadioptric sensor, again using only lines. Let us gain some intuition as to why this is true, and why it is not possible to calibrate a normal perspective camera with these simple assumptions.

First examine the perspective case. Assuming that aspect ratio is one, there are two intrinsic parameters, namely the image center and focal length. The image of a line in space is a line in the image plane, and any given line may be uniquely determined by two points. From any image line it is possible only to determine the orientation of the plane containing the line in space and the focal point; the orientation of this plane can be parameterized by two parameters. Given n lines, how many constraints are there and how many unknowns? If for some n the number of constraints exceeds the number of unknowns, then we have a hope of obtaining the unknowns, and thus calibrate the sensor. However, for every line added we gain two more constraints and two more unknowns; we will always be short by three equations. Therefore self-calibration *from lines*, without any metric information, and in one frame is hopeless in the perspective case.

What about the parabolic case? There are a total of three unknowns, focal length and image center (alone giving two unknowns). The projection of any line is a circle, and which is completely specified by as few as three points, therefore three constraints. The orientation of the plane containing the line gives two unknowns. So, for every line that we obtain we reduce the number of unknowns by one. If there are three lines, we have 9 constraints and 9 unknowns, and thus we can perform self-calibration with only three lines.

Finally the hyperbolic case. There are four unknowns and each line adds two for orientation. The projection of a line is a conic which may be specified by five points. Thus when we have two lines we have 8 unknowns and 10 constraints. So, with only two lines the system is over-determined, but nevertheless we can still perform a calibration.

We give here a simple and compact algorithm for calibrating the parabolic projection. It is based on the fact that a sphere whose equator is an image line in the image plane contains the point $(c_x, c_y, 2p)$, where (c_x, c_y) is assumed to be the image center, though initially unknown. This is by symmetry, since the image circle intersects the fronto-parallel plane at points a distance $2p$ from the image center. Thus the intersection of at least three spheres so-constructed will produce the points $(c_x, c_y, \pm 2p)$, giving us both image center and focal length simultaneously.

In the presence of noise, the intersection will not be defined for more than three spheres, yet we may minimize the distance from a point to all of the spheres, i.e. find the point (c_x, c_y, p) such that

$$\sum_{i=1}^{n} \left((d_x^i - c_x)^2 + (d_y^i - c_y)^2 + 4p^2 - r_i^2 \right)^2 \qquad (1)$$

is a minimum over all points. Here (d_x^i, d_y^i) is the center of the i-th image circle, and r_i is its radius. The intersection is not defined for fewer than three spheres, since the intersection of two spheres gives only the circle within which the point lies, but not the point itself.

4 Experiment

We present here a short experiment with real data as a proof of concept. We will show how given a single catadioptric image of a plane (left image in Fig. 6) we can recover the intrinsic parameters of the camera and metrically rectify this plane, too. The system used is an off-shelf realization (S1 model, Cyclovision Inc.) of a parabolic catadioptric system invented by Nayar [8]. The algorithm detects edge points and groups them in elliptical arcs using a Delaunay triangulation of the points and a subsequent Hough transform. An ellipse fitting algorithm [3] is then applied on the clustered points. The aspect ratio is eliminated and the ellipses are transformed to circles (Fig. 6, middle). We additionally assume that these lines are coplanar and that they belong to two groups of parallel lines. However, we do not make any assumption about the angles between these lines.

From the parallelism assumption we know that the intersections of the circles are the antipodal projections of vanishing points. The calibration theory developed above tells us that the intersection of the lines connecting the antipodal points gives the image center.

Two vanishing points and the focal point define a plane parallel to the plane viewed. Imagine the horizon of this plane (line at infinity) defined by the two sets of vanishing points. Imagine also a pole on the sphere corresponding to the plane spanned by the horizon and the focal point. The parabolic projection of the horizon is a circle (the horizon conic) and its center is the projection of the pole. However, this pole gives exactly the normal where all the lines lie. This center is the dual point to the line which is the horizon of the perspective projection. Given the calculation (1) the focal length is directly obtained. This focal length is the effective focal length required for any operation in the catadioptric system (we can not decouple the mirror from the lens focal length). We have thus been able to compute image center, focal length, and the normal of a plane without assuming any metric information. We visualize the result on the right of Fig. 6 where we have rectified the ceiling plane so that it looks as if it were fronto-parallel. Unlike the planar perspective case [6] metric rectification of a plane from a single image is possible with a parabolic catadioptric system without any metric information.

5 Conclusion

In this paper, we presented a novel theory on the geometry of central panoramic or catadioptric vision systems. We proved that every projection can be modeled with the projection of the sphere to a horizontal plane from a point on the vertical axis of the sphere. This modeling includes traditional cameras which

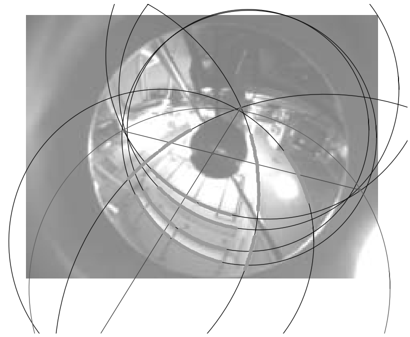

Fig. 6. Top left: Original image of the ceiling recorded by the catadioptric camera slanted approx. 45 deg. with respect to the ceiling. Top right: Two groups of four and three circles, respectively, fitted on the images of the ceiling-edges. The lines through the vanishing points intersect at the image center and all the vanishing points lie on a circle. Bottom middle: Both, the collinearity of the edge elements and the perpendicularity of the edges show a superior performance in estimating intrinsics as well as pan-tilt of the ceiling using only natural landmarks.

are equivalent to a catadioptric projection via a planar mirror. In this case the projection point of our model is the center of the sphere. In the parabolic case, the projection point becomes the north-pole and the projection is a stereography. The conformal mapping properties of the stereography show the power of the parabolic systems. Hyperbolic or elliptical mirrors correspond to projections from points on the vertical diameter within the sphere. We showed that projections of point and lines in space are points and conics, respectively. Due to preservation of the incidence relationship we can regard the conics as projective lines. We showed that these projective lines are indeed dual to the points and vice versa.

Very useful practical implications can be directly derived from this theory. Calibration constraints are natural and we provided a geometric argument why all catadioptric systems except the conventional planar projection can be calibrated from one view. We gave an experimental evidence using a parabolic mirror where we also showed that metric rectification of a plane is possible if we have only affine but not metric information about the environment. We plan to extend our theory to multiple catadioptric views as well as to the study of robustness of scene recovery using the above principles.

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